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New oscillation criteria for linear matrix Hamiltonian systems

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Abstract

Some new oscillation criteria are established for the matrix linear Hamiltonian system $X' = A(t)X + B(t)Y$, $Y' = C(t)X - A^*(t)Y$ under the hypothesis: $A(t)$, $B(t) = B^*(t) > 0$, and $C(t) = C^*(t)$ are $n \times n$ real continuous matrix functions on the interval $[t_0, \infty)$, $(-\infty < t_0)$. These results are sharper than some previous results even for self-adjoint second order matrix differential systems.

Keywords: Oscillation; Linear; Hamiltonian systems

1. Introduction

Consider the matrix linear Hamiltonian system

$$\begin{cases} X' = A(t)X + B(t)Y, \\ Y' = C(t)X - A^*(t)Y, \end{cases} \quad t \geqslant t_0, \tag{1.1}$$

where $A(t), B(t), C(t)$ are $(n \times n)$-matrices, and B, C are Hermitian, i.e., $B^*(t) = B(t)$, $C^*(t) = C(t)$. By M^* we mean the conjugate transpose of the matrix M.

A Hermitian matrix $M \in C^{n \times n}$ is positive semidefinite (positive definite) if for all $u \in C^n$, $u \neq 0$, $u^*Mu \geqslant 0$ (> 0). A positive semidefinite (positive definite) Hermitian matrix M will be denoted by $M \geqslant 0$ $(M > 0)$, and the usual ordering of the eigenvalues of M by $\lambda_1[M] \geqslant \lambda_2[M] \geqslant \cdots \geqslant \lambda_n[M]$.

For any two solutions X_1, Y_1 and X_2, Y_2 of (1.1), the Wronskian $X_1^*(t)Y_2(t) - Y_1^*(t)X_2(t)$ is a constant matrix. In particular, for any solution X, Y of (1.1), $X^*(t)Y(t) -$

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$Y^*(t)X(t)$ is a constant matrix. The solution X, Y of (1.1) is said to be conjoined if $X^*(t)Y(t) - Y^*(t)X(t) = 0$. A conjoined solution X, Y of (1.1) is said to be a conjoined basis of (1.1) if the rank of the $(2n \times n)$-matrix $(X(t), Y(t))$ is n. A conjoined basis X and Y of (1.1) is said to be oscillatory on $[t_0, \infty)$ if $\det X(t)$ has arbitrarily large zeros. System (1.1) is said to be oscillatory on $[t_0, \infty)$ if one conjoined basis of (1.1) is oscillatory.

In the case when $A(t) \equiv 0$, $B(t) > 0$, (1.1) reduces to the second-order self-adjoint matrix differential system

$$\big(P(t)X'\big)' + Q(t)X = 0 \tag{1.2}$$

with $P(t) = B^{-1}(t)$, $Q(t) = -C(t)$. The oscillation and nonoscillation of (1.1) or (1.2) have been extensively studied by many authors [1–13]. A discrete version of (1.2) is studied in [14]. However, all these results are given in the form of $\lim_{t\to\infty} \sup \lambda_1[\cdot] = +\infty$. In this paper we establish some new oscillation criteria which are presented in the form of $\lim_{t\to\infty} \sup \lambda_1[\cdot] >$ const for the system (1.1) by using one particular function $\Phi(t, s, r)$ defined as

$$\Phi(t, s, r) = (t - s)^\alpha (s - r)^\beta, \quad \text{where } \alpha, \beta > \frac{1}{2} \text{ are constants and } r \geqslant t_0. \tag{1.3}$$

Our results improve most known oscillation results even for the self-adjoint differential system (1.2). This can be seen by the examples given at the end of this paper.

2. Main results

Let $\phi(t)$ and $\theta(t)$ be positive, smooth and real-valued functions on $[t_0, \infty)$. Suppose that $B(t) > 0$. Let us make a change of unknown variables

$$U = \phi X, \qquad V = \theta Y + \alpha B^{-1} X,$$

where

$$\alpha = \frac{\theta}{2}\left(\frac{\phi'}{\phi} - \frac{\theta'}{\theta}\right).$$

Then U and V satisfy a differential system

$$\begin{cases} U' = A(t)U + B_1(t)V + \frac{1}{2}\big(\frac{\phi'}{\phi} + \frac{\theta'}{\theta}\big)U, \\ V' = C_1(t)U - A^*(t)V + \frac{1}{2}\big(\frac{\phi'}{\phi} + \frac{\theta'}{\theta}\big)V, \end{cases} \tag{2.1}$$

where

$$B_1(t) = \frac{\phi(t)}{\theta(t)} B(t), \tag{2.2}$$

$$C_1(t) = \frac{\theta}{\phi}\left\{C(t) + \frac{\alpha}{\theta}\big(B^{-1}(t)A(t) + A^*(t)B^{-1}(t)\big) + \left(\frac{\alpha}{\theta}B^{-1}(t)\right)' - \frac{\alpha^2}{\theta^2}B^{-1}(t)\right\}. \tag{2.3}$$

Now we give the main results of this paper.

Theorem 1. *Let* $\Phi(t,s,r)$ *be defined by* (1.3). *If there exist two positive and real-valued functions* $\phi,\theta\in C^1[t_0,\infty)$ *such that for every* $r\geqslant t_0$

$$\lim_{t\to\infty}\sup\lambda_1\left[\int_r^t M_1(t,s,r)\,ds\right]>0, \tag{2.4}$$

where

$$M_1(t,s,r)=\Phi^2(t,s,r)D_1(s)+\Phi_s'(t,s,r)\Phi(t,s,r)K_1(s)-\big(\Phi_s'(t,s,r)\big)^2B_1^{-1}(s), \tag{2.5}$$

$$D_1(t)=-C_1(t)-\big(A^*B_1^{-1}A\big)(t),\qquad K_1(t)=\big(A^*B_1^{-1}+B_1^{-1}A\big)(t), \tag{2.6}$$

then (1.1) *is oscillatory.*

Proof. Assume to the contrary that (1.1) is nonoscillatory. Then $X(t)$ is nonsingular for all sufficiently large t, say $t\geqslant T\geqslant t_0$, for any conjoined basis $X(t),Y(t)$ of (1.1). This allows us to make a transformation $W(t)=V(t)U^{-1}(t)$, $t\geqslant T$. From (2.1) we have

$$\begin{aligned}W'(t)&=-(A^*W+WA+WB_1W-C_1)(t)\\&=-\big\{\big(W+B_1^{-1}A\big)^*B_1\big(W+B_1^{-1}A\big)\big\}(t)-D_1(t).\end{aligned} \tag{2.7}$$

Multiplying (2.7), with t replaced by s, by $\Phi^2(t,s,T)$ and integrating from T to t, we obtain

$$\begin{aligned}\int_T^t\Phi^2(t,s,T)D_1(s)\,ds&=-\int_T^t\Phi^2(t,s,T)W'(s)\,ds\\&\quad-\int_T^t\Phi^2(t,s,T)\big\{\big(W+B_1^{-1}A\big)^*B_1\big(W+B_1^{-1}A\big)\big\}(s)\,ds\\&=2\int_T^t\Phi(t,s,T)\Phi_s'(t,s,T)W(s)\,ds\\&\quad-\int_T^t\Phi^2(t,s,T)\big\{\big(W+B_1^{-1}A\big)^*B_1\big(W+B_1^{-1}A\big)\big\}(s)\,ds.\end{aligned} \tag{2.8}$$

According to the direct computation, we see that

$$\begin{aligned}&2\int_T^t\Phi(t,s,T)\Phi_s'(t,s,T)W(s)\,ds\\&\quad-\int_T^t\Phi^2(t,s,T)\big\{\big(W+B_1^{-1}A\big)^*B_1\big(W+B_1^{-1}A\big)\big\}(s)\,ds\end{aligned}$$

$$
\begin{aligned}
&= -R_1^{-1}(s)Q_1^*(t,s,T)Q_1(t,s,T)R_1^{-1}(s) \\
&\quad - \Phi(t,s,T)\Phi_s'(t,s,T)K_1(s) + \big(\Phi_s'(t,s,T)\big)^2 B_1^{-1}(s),
\end{aligned}
$$

where

$$
\begin{aligned}
&R_1(s) = B_1^{1/2}(s), \\
&Q_1(t,s,T) = \Phi(t,s,T)\big\{R_1\big(W + B_1^{-1}A\big)R_1\big\}(s) - \Phi_s'(t,s,T)I_n.
\end{aligned}
$$

Thus from (2.8) and the above computation, we have

$$
\begin{aligned}
&\int_T^t \Phi^2(t,s,T)D_1(s)\,ds \\
&\quad = -\int_T^t R_1^{-1}(s)Q_1^*(t,s,T)Q_1(t,s,T)R_1^{-1}(s)\,ds \\
&\qquad - \int_T^t \big[\Phi(t,s,T)\Phi_s'(t,s,T)K_1(s) - \big(\Phi_s'(t,s,T)\big)^2 B_1^{-1}(s)\big]\,ds. \qquad (2.9)
\end{aligned}
$$

Therefore, from (2.5), (2.6), and (2.9), for $t \geqslant T$ we obtain

$$
\int_T^t M_1(t,s,T)\,ds = -\int_T^t R_1^{-1}(s)Q_1^*(t,s,T)Q_1(t,s,T)R_1^{-1}(s)\,ds \leqslant 0. \qquad (2.10)
$$

This implies that

$$
\limsup_{t\to\infty} \lambda_1\left[\int_T^t M_1(t,s,T)\,ds\right] \leqslant 0,
$$

which contradicts (2.4). This completes the proof of Theorem 1. □

Remark. We observe that only the ratios α/θ and ϕ/θ are involved in the coefficients of (2.2) and (2.3), and $\alpha(t) = \theta\{(1/2)\log[\phi(t)/\theta(t)]\}'$. Therefore, if we put

$$
a(t) = \frac{\theta(t)}{\phi(t)} = \exp\left(-2\int_{t_0}^t f(s)\,ds\right),
$$

where $f \in C^1[t_0,\infty)$, then

$$
\frac{\alpha(t)}{\theta(t)} = f(t), \qquad \frac{\alpha(t)}{\phi(t)} = a(t)f(t),
$$

and hence

$$
\begin{aligned}
&B_1(t) = \frac{1}{a(t)}B(t), \\
&C_1(t) = a(t)\big[C(t) + f(t)(B^{-1}A + A^*B^{-1})(t) + \big(f(t)B^{-1}(t)\big)' - f^2(t)B^{-1}(t)\big].
\end{aligned}
$$

In other words, to carry out the transformation, we need only to choose one appropriate smooth function $f(t)$.

If we choose appropriate θ and ϕ in Theorem 1 such that $B_1^{-1}(t) = (\theta/\phi)B^{-1}(t) \leqslant I_n$ for $t \geqslant t_0$, i.e., $a(t)B^{-1}(t) \leqslant I_n$ for $t \geqslant t_0$, and let $\Phi(t,s,r) = (t-s)(s-r)^\alpha$ for $\alpha > 1/2$, then we have the following theorem from Theorem 1.

Theorem 2. *System* (1.1) *is oscillatory provided that for some* $\alpha > 1/2$ *and for every* $r \geqslant t_0$

$$\limsup_{t\to\infty} \frac{1}{t^{2\alpha+1}} \lambda_1 \left[\int_r^t (t-s)^2 (s-r)^{2\alpha} \left(D_1(s) + \frac{\alpha t - (\alpha+1)s + r}{(t-s)(s-r)} K_1(s) \right) ds \right] > \frac{\alpha}{(2\alpha-1)(2\alpha+1)}, \tag{2.11}$$

where $D_1(s)$, $K_1(s)$ *are defined as in Theorem* 1.

Proof. Assume to the contrary that (1.1) is nonoscillatory. Then $X(t)$ is nonsingular for all sufficiently large t, say $t \geqslant T \geqslant t_0$. Similar to the proof of Theorem 1 and because of $B_1^{-1} \leqslant I_n$, for $t \geqslant T \geqslant t_0$ we have

$$\begin{aligned} &\int_T^t \left\{ (t-s)^2 (s-T)^{2\alpha} D_1(s) + (t-s)(s-T)^{2\alpha-1} \big[\alpha t - (\alpha+1)s + T \big] K_1(s) \right\} ds \\ &\quad \leqslant \int_T^t \big[\alpha(t-s)(s-T)^{\alpha-1} - (s-T)^\alpha \big]^2 B_1^{-1}(s)\, ds \\ &\quad \leqslant \int_T^t \big[\alpha(t-s)(s-T)^{\alpha-1} - (s-T)^\alpha \big]^2 I_n\, ds. \end{aligned}$$

This implies that

$$\lambda_1 \left[\int_T^t (t-s)^2 (s-T)^{2\alpha} \left(D_1(s) + \frac{\alpha t - (\alpha+1)s + T}{(t-s)(s-T)} K_1(s) \right) ds \right] \leqslant \int_T^t \big[\alpha(t-s)(s-T)^{\alpha-1} - (s-T)^\alpha \big]^2 ds. \tag{2.12}$$

Note that for any $T \geqslant t_0$

$$\int_T^t \big[\alpha(t-s)(s-T)^{\alpha-1} - (s-T)^\alpha \big]^2 ds = \frac{\alpha}{(2\alpha-1)(2\alpha+1)} (t-T)^{2\alpha+1}. \tag{2.13}$$

Thus, from (2.12) and (2.13) we obtain that

$$\lambda_1\left[\int_T^t (t-s)^2(s-T)^{2\alpha}\left(D_1(s)+\frac{\alpha t-(\alpha+1)s+T}{(t-s)(s-T)}K_1(s)\right)ds\right]$$
$$\leqslant \frac{\alpha}{(2\alpha-1)(2\alpha+1)}(t-T)^{2\alpha+1}.$$

It follows that

$$\lim_{t\to\infty}\sup\frac{1}{t^{2\alpha+1}}\lambda_1\left[\int_r^t (t-s)^2(s-T)^{2\alpha}\left(D_1(s)+\frac{\alpha t-(\alpha+1)s+T}{(t-s)(s-T)}K_1(s)\right)ds\right]$$
$$\leqslant \frac{\alpha}{(2\alpha-1)(2\alpha+1)},$$

which contradicts assumption (2.11). This completes the proof of Theorem 2. □

If we choose $\Phi(t,s,r)=(t-s)^\alpha(s-r)$ for $\alpha>1/2$, similar to the proof of Theorem 2, we can easily obtain the following theorem.

Theorem 3. *System* (1.1) *is oscillatory provided that for some $\alpha>1/2$ and for every $r\geqslant t_0$*

$$\lim_{t\to\infty}\sup\frac{1}{t^{2\alpha+1}}\lambda_1\left[\int_r^t (t-s)^{2\alpha}(s-r)^2\left(D_1(s)+\frac{t-(\alpha+1)s+\alpha r}{(t-s)(s-r)}K_1(s)\right)ds\right]$$
$$> \frac{\alpha}{(2\alpha-1)(2\alpha+1)}, \tag{2.14}$$

where $D_1(s)$, $K_1(s)$ are defined as in Theorem 1.

Proof. Assume to the contrary that (1.1) is nonoscillatory. Then $X(t)$ is nonsingular for all sufficiently large t, say $t\geqslant T\geqslant t_0$. Similar to the proof of Theorem 2 and because of $B_1^{-1}\leqslant I_n$, for $t\geqslant T\geqslant t_0$ we have

$$\lambda_1\left[\int_T^t (t-s)^{2\alpha}(s-T)^2\left(D_1(s)+\frac{t-(\alpha+1)s+\alpha T}{(t-s)(s-T)}K_1(s)\right)ds\right]$$
$$\leqslant \int_T^t \left[(t-s)^\alpha-\alpha(t-s)^{\alpha-1}(s-T)\right]^2 ds.$$

Noting that

$$\int_T^t \left[(t-s)^\alpha-\alpha(t-s)^{\alpha-1}(s-T)\right]^2 ds=\frac{\alpha}{(2\alpha-1)(2\alpha+1)}(t-T)^{2\alpha+1},$$

we obtain a contradiction that

$$\lim_{t\to\infty}\sup\frac{1}{t^{2\alpha+1}}\lambda_1\left[\int_T^t (t-s)^{2\alpha}(s-T)^2\left(D_1(s)+\frac{t-(\alpha+1)s+\alpha T}{(t-s)(s-T)}K_1(s)\right)ds\right]$$
$$\leqslant \frac{\alpha}{(2\alpha-1)(2\alpha+1)}.$$

This completes the proof of Theorem 3. $\square$

In order to show the sharpness of our results, now let us consider the following two examples:

Example 1. Consider the Euler differential system

$$Y''+\mathrm{diag}\left(\frac{\beta}{t^2},\frac{\gamma}{t^2}\right)Y=0,\quad t\geqslant 1,\ \gamma\geqslant\beta>0. \tag{2.15}$$

If we choose $f(t)=0$, then $a(t)=1$, $D_1(t)=\mathrm{diag}(\beta/t^2,\gamma/t^2)$ and $K_1(t)=0$. Note that for each $r\geqslant t_0$

$$\lim_{t\to\infty}\frac{1}{t^{2\alpha+1}}\int_r^t (t-s)^2(s-r)^{2\alpha}\frac{\gamma}{s^2}\,ds=\frac{\gamma}{\alpha(2\alpha-1)(2\alpha+1)}.$$

For any $\gamma>1/4$, there exists $\alpha>1/2$ such that

$$\frac{\gamma}{\alpha(2\alpha-1)(2\alpha+1)}>\frac{\alpha}{(2\alpha-1)(2\alpha+1)},$$

i.e., $\gamma>\alpha^2$. This means that (2.11) holds. By Theorem 2, we find that system (2.15) is oscillatory for $\gamma>1/4$. However, we can easily show that criteria in [3,6,8] fail to reveal this fact.

Example 2. Consider the 4-dimensional system (1.1) with $t\geqslant 1$ and

$$A(t)=\begin{bmatrix}0 & -1/t\\ 2/t & 0\end{bmatrix},\qquad B(t)=\begin{bmatrix}t & 0\\ 0 & 2t\end{bmatrix},\qquad C(t)=-\begin{bmatrix}\theta/t^3 & 0\\ 0 & \eta/t^3\end{bmatrix},$$

where $\eta\geqslant\theta>0$ and $t\geqslant 1$. According to the remark, if we let $f(t)=-1/2t$, then $a(t)=t$ and $a(t)B^{-1}(t)\leqslant I_2$ for $t\geqslant 1$. Thus, from (2.6) we have

$$K_1(t)\equiv 0,\qquad D_1(t)=\begin{bmatrix}(\theta-11/4)t^{-2} & 0\\ 0 & (\eta-11/8)t^{-2}\end{bmatrix}.$$

Similar to the proof of Example 1, we can obtain that system (1.1) is oscillatory when $\eta>13/8$ by Theorem 2.

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